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Differential equations with integral boundary conditions

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Abstract

The method of lower and upper solutions combined with monotone iterative technique is used for ordinary differential equations with integral boundary conditions. Problems of existence of extremal and unique solutions are discussed. Some comparison results are formulated too. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we shall consider the following differential problem:

$$x'(t) = f(t, x(t)), \quad t \in J = [0, T], \ T > 0,$$

$$x(0) = \lambda \int_0^T x(s)\,\mathrm{d}s + d, \quad d \in \mathbb{R}, \tag{1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R})$ and $\lambda = 1$ or -1. Note that problems with integral conditions are discussed in many papers, see for example [6,7], where a numerical–analytic method is applied.

It is well known that the monotone iterative technique is a powerful method used to approximate solutions of several problems (see, for example, [1–5,8,9]). The purpose of this paper is to show that it can be applied successfully to problems with integral boundary conditions of type (1). This technique play important roles in constructing monotone sequences which converge to solutions of our problems. A one-sided Lipschitz condition imposed on f (with respect to the second variable) guarantees that problem (1) with $\lambda = 1$ has extremal solutions (see Section 2). For case $\lambda = -1$, we need to add extra condition on f_x to show that problem (1) has a unique solution (see Section 3). Some examples are given to illustrate obtained results.

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2. Case $\lambda = 1$

A function $u \in C^1(J, \mathbb{R})$ is said to be a lower solution of problem (1) for $\lambda = 1$ if

$$u'(t) \leqslant f(t, u(t)), \quad t \in J,$$

$$u(0) \leqslant \int_0^T u(s)\,\mathrm{d}s + d,$$

and an upper solution of (1) if the inequalities are reversed.

Let $\Omega = \{u\colon y_0(t) \leqslant u \leqslant z_0(t),\ t \in J\}$ and $\Delta = \{w \in C^1(J, \mathbb{R})\colon y_0(t) \leqslant w(t) \leqslant z_0(t),\ t \in J\}$ be non-empty sets.

We introduce the following assumptions for later use.

(H_1) $f \in C(J \times \Omega, \mathbb{R})$,
(H_2) $y_0, z_0 \in C^1(J, \mathbb{R})$ are lower and upper solutions of (1) for $\lambda = 1$, respectively, and such that $y_0(t) \leqslant z_0(t)$, $t \in J$,
(H_3) there exists $M > 0$ such that $f(t, u) - f(t, v) \leqslant M[v - u]$ if $u \leqslant v$, $u, v \in \Omega$, $t \in J$.

Lemma 1. *Put $\lambda = 1$. Let Assumptions H_1 and H_3 hold. Assume that $u, v \in \Delta$ are lower and upper solutions of problem (1), respectively, and $u(t) \leqslant v(t)$ on J. If*

$$y'(t) = f(t, u(t)) - M[y(t) - u(t)], \ t \in J, \quad y(0) = \int_0^T u(s)\,\mathrm{d}s + d,$$

$$z'(t) = f(t, v(t)) - M[z(t) - v(t)], \ t \in J, \quad z(0) = \int_0^T v(s)\,\mathrm{d}s + d,$$

then

$$u(t) \leqslant y(t) \leqslant z(t) \leqslant v(t), \quad t \in J, \tag{2}$$

and y, z are lower and upper solutions of (1), respectively.

Proof. Note that there exist unique solutions for y and z. Put $p = u - y$, $q = z - v$, so

$$p(0) \leqslant \int_0^T u(s)\,\mathrm{d}s - \int_0^T u(s)\,\mathrm{d}s = 0, \quad q(0) \leqslant \int_0^T v(s)\,\mathrm{d}s - \int_0^T v(s)\,\mathrm{d}s = 0,$$

and

$$p'(t) \leqslant f(t, u(t)) - f(t, u(t)) + M[y(t) - u(t)] = -Mp(t), \quad t \in J,$$

$$q'(t) \leqslant f(t, v(t)) - M[z(t) - v(t)] - f(t, v(t)) = -Mq(t), \quad t \in J.$$

Hence $p(t) \leqslant e^{-Mt}p(0) \leqslant 0$, $q(t) \leqslant e^{-Mt}q(0) \leqslant 0$, $t \in J$ showing that $u(t) \leqslant y(t)$, $z(t) \leqslant v(t)$, $t \in J$. Now let $p = y - z$, so $p(0) = \int_0^T u(s)\,ds - \int_0^T v(s)\,ds \leqslant 0$. Assumption H_3 yields

$$\begin{aligned} p'(t) &\leqslant f(t,u(t)) - f(t,v(t)) - M[y(t) - u(t) - z(t) + v(t)] \\ &\leqslant M[v(t) - u(t)] - M[p(t) + v(t) - u(t)] = -Mp(t), \quad t \in J. \end{aligned}$$

Hence $p(t) \leqslant 0$, $t \in J$ showing that $y(t) \leqslant z(t)$, $t \in J$. It proves that (2) holds. Now we need to show that y, z are lower and upper solutions of (1), respectively. Using again Assumption H_3 we see

$$\begin{aligned} y'(t) &= f(t,u(t)) - M[y(t) - u(t)] - f(t,y(t)) + f(t,y(t)) \\ &\leqslant f(t,y(t)) + M[y(t) - u(t)] - M[y(t) - u(t)] = f(t,y(t)), \quad t \in J, \end{aligned}$$

$$\begin{aligned} z'(t) &= f(t,v(t)) - M[z(t) - v(t)] - f(t,z(t)) + f(t,z(t)) \\ &\geqslant f(t,z(t)) - M[v(t) - z(t)] - M[z(t) - v(t)] = f(t,z(t)), \quad t \in J. \end{aligned}$$

and

$$y(0) = \int_0^T u(s)\,ds + d \leqslant \int_0^T y(s)\,ds + d, \quad z(0) = \int_0^T v(s)\,ds + d \geqslant \int_0^T z(s)\,ds + d.$$

It shows that y, z are lower and upper solutions of (1), respectively. It ends the proof. □

Theorem 1. *Put $\lambda = 1$. Let Assumptions H_1, H_2 and H_3 hold. Then there exist monotone sequences $\{y_n, z_n\}$ such that $y_n(t) \to y(t)$, $z_n(t) \to z(t)$, $t \in J$ as $n \to \infty$ and this convergence is uniformly and monotonically on J. Moreover, y, z are extremal solutions of (1) in Δ.*

Proof. Let

$$y'_{n+1}(t) = f(t, y_n(t)) - M[y_{n+1}(t) - y_n(t)], \ t \in J, \quad y_{n+1}(0) = \int_0^T y_n(s)\,ds + d,$$

$$z'_{n+1}(t) = f(t, z_n(t)) - M[z_{n+1}(t) - z_n(t)], \ t \in J, \quad z_{n+1}(0) = \int_0^T z_n(s)\,ds + d$$

for $n = 0, 1, \ldots$. Lemma 1 shows $y_0(t) \leqslant y_1(t) \leqslant z_1(t) \leqslant z_0(t)$, $t \in J$, and y_1, z_1 are lower and upper solutions of (1), respectively.

Assume that

$$y_0(t) \leqslant y_1(t) \leqslant \cdots \leqslant y_k(t) \leqslant z_k(t) \leqslant \cdots \leqslant z_1(t) \leqslant z_0(t), \quad t \in J$$

for some $k \geqslant 1$ and let y_k, z_k be lower and upper solutions of (1), respectively. Then, using again Lemma 1, we get $y_k(t) \leqslant y_{k+1}(t) \leqslant z_{k+1}(t) \leqslant z_k(t)$, $t \in J$. By induction, we have

$$y_0(t) \leqslant y_1(t) \leqslant \cdots \leqslant y_n(t) \leqslant z_n(t) \leqslant \cdots \leqslant z_1(t) \leqslant z_0(t), \quad t \in J$$

for all n. Hence $y_n(t) \to y(t)$, $z_n(t) \to z(t)$, $t \in J$ if $n \to \infty$. Indeed, y and z are solutions of problem (1).

To finish the proof it is enough to show that y and z are extremal solutions of (1) in Δ. To do it, we need to show that if w is any solution of (1) such that $y_0(t) \leqslant w(t) \leqslant z_0(t)$, $t \in J$, then $y_0(t) \leqslant y(t) \leqslant w(t) \leqslant z(t) \leqslant z_0(t)$, $t \in J$. Assume that for some k, $y_k(t) \leqslant w(t) \leqslant z_k(t)$, $t \in J$. Put $p = y_{k+1} - w$, $q = w - z_{k+1}$. Then

$$p(0) = \int_0^T [y_k(s) - w(s)]\,\mathrm{d}s \leqslant 0, \quad q(0) = \int_0^T [w(s) - z_k(s)]\,\mathrm{d}s \leqslant 0,$$

and

$$\begin{aligned} p'(t) &= f(t, y_k(t)) - f(t, w(t)) - M[y_{k+1}(t) - y_k(t)] \\ &\leqslant M[w(t) - y_k(t)] - M[y_{k+1}(t) - y_k(t)] = -Mp(t), \quad t \in J, \\ q'(t) &= f(t, w(t)) - f(t, z_k(t)) + M[z_{k+1}(t) - z_k(t)] \\ &\leqslant M[z_k(t) - w(t)] + M[z_{k+1}(t) - z_k(t)] = -Mq(t), \quad t \in J. \end{aligned}$$

Hence, $y_{k+1}(t) \leqslant w(t) \leqslant z_{k+1}(t)$, $t \in J$. It proves, by induction, that $y_n(t) \leqslant w(t) \leqslant z_n(t)$, $t \in J$ for all n. Taking the limit $n \to \infty$, we get $y(t) \leqslant w(t) \leqslant z(t)$ on J so the assertion of Theorem 1 is true. The proof is complete. □

Example 1. Consider the following problem:

$$\text{(a)} \quad \begin{aligned} &x'(t) = \mathrm{e}^{t \sin^2 x(t)}, \quad t \in J = [0, T] \text{ with } T = \ln 2, \\ &x(0) = \int_0^T x(s)\,\mathrm{d}s. \end{aligned}$$

Indeed, $0 < \mathrm{e}^{t \sin^2 x} \leqslant \mathrm{e}^t$, $t \in J$, $x \in \mathbb{R}$. Note that $y_0(t) = 0$, $z_0(t) = \mathrm{e}^t$ on J are lower and upper solutions of problem (a), respectively. Moreover, $M = 2\ln 2$. By Theorem 1, problem (a) has extremal solutions in the segment $[y_0, z_0]$.

3. Case $\lambda = -1$

Functions $u, v \in C^1(J, \mathbb{R})$ are called weakly coupled (w.c.) lower and upper solutions of problem (1) for $\lambda = -1$ if

$$u'(t) \leqslant f(t, u(t)), \ t \in J, \quad u(0) \leqslant -\int_0^T v(s)\,\mathrm{d}s + d,$$

$$v'(t) \geqslant f(t, v(t)), \ t \in J, \quad v(0) \geqslant -\int_0^T u(s)\,\mathrm{d}s + d.$$

Example 2. Consider the following problem:

$$\text{(b)} \quad \begin{aligned} &x'(t) = \mathrm{e}^{t \sin^2 x(t)}, \quad t \in J = [0, T] \text{ with } T = \ln 2, \\ &x(0) = -\int_0^T x(s)\,\mathrm{d}s. \end{aligned}$$

It is simple to verify that $y_0(t) = -1$, $z_0(t) = \mathrm{e}^t$ on J are w.c. lower and upper solutions of (b).

We introduce the following assumptions for later use.

(H_4) $y_0, z_0 \in C^1(J, \mathbb{R})$ are w.c. lower and upper solutions of (1) for $\lambda = -1$, and such that $y_0(t) \leqslant z_0(t)$, $t \in J$,

(H_5) $f_x \in C(J \times \Omega, \mathbb{R})$ and $\int_0^T \mathrm{e}^{\int_0^t f_x(s,\xi(s))\,\mathrm{d}s}\,\mathrm{d}t \neq 1$ for any $\xi \in \Omega$.

The sets Ω and Δ are defined as in Section 2.

Lemma 2. *Put* $\lambda = -1$. *Let Assumptions* H_1, H_3 *hold. Assume that* $u, v \in \Delta$ *are w.c. lower and upper solutions of problem* (1), *and* $u(t) \leqslant v(t)$ *on* J. *If*

$$y'(t) = f(t, u(t)) - M[y(t) - u(t)], \quad t \in J, \quad y(0) = -\int_0^T v(s)\,\mathrm{d}s + d,$$

$$z'(t) = f(t, v(t)) - M[z(t) - v(t)], \quad t \in J, \quad z(0) = -\int_0^T u(s)\,\mathrm{d}s + d,$$

then

$$u(t) \leqslant y(t) \leqslant z(t) \leqslant v(t), \quad t \in J, \tag{3}$$

and y, z *are w.c. lower and upper solutions of* (1).

Proof. Note that there exist unique solutions for y and z. Put $p = u - y$, $q = z - v$, so

$$p(0) \leqslant -\int_0^T v(s)\,\mathrm{d}s + \int_0^T v(s)\,\mathrm{d}s = 0, \quad q(0) \leqslant -\int_0^T u(s)\,\mathrm{d}s + \int_0^T u(s)\,\mathrm{d}s = 0,$$

and

$$p'(t) \leqslant f(t, u(t)) - f(t, u(t)) + M[y(t) - u(t)] = -Mp(t), \quad t \in J,$$

$$q'(t) \leqslant f(t, v(t)) - M[z(t) - v(t)] - f(t, v(t)) = -Mq(t), \quad t \in J.$$

Hence $p(t) \leqslant \mathrm{e}^{-Mt} p(0) \leqslant 0$, $q(t) \leqslant \mathrm{e}^{-Mt} q(0) \leqslant 0$, $t \in J$ showing that $u(t) \leqslant y(t)$, $z(t) \leqslant v(t)$, $t \in J$. Now let $p = y - z$, so $p(0) = -\int_0^T v(s)\,\mathrm{d}s + \int_0^T u(s)\,\mathrm{d}s \leqslant 0$. Assumption H_3 yields

$$\begin{aligned} p'(t) &\leqslant f(t, u(t)) - f(t, v(t)) - M[y(t) - u(t) - z(t) + v(t)] \\ &\leqslant M[v(t) - u(t)] - M[p(t) + v(t) - u(t)] = -Mp(t), \quad t \in J. \end{aligned}$$

Hence $p(t) \leqslant 0$, $t \in J$ showing that $y(t) \leqslant z(t)$, $t \in J$. It proves that (3) holds. Now we need to show that y, z are w.c. lower and upper solutions of (1). Using again Assumption H_3 we obtain

$$\begin{aligned} y'(t) &= f(t, u(t)) - M[y(t) - u(t)] - f(t, y(t)) + f(t, y(t)) \\ &\leqslant f(t, y(t)) + M[y(t) - u(t)] - M[y(t) - u(t)] = f(t, y(t)), \quad t \in J, \end{aligned}$$

$$\begin{aligned} z'(t) &= f(t, v(t)) - M[z(t) - v(t)] - f(t, z(t)) + f(t, z(t)) \\ &\geqslant f(t, z(t)) - M[v(t) - z(t)] - M[z(t) - v(t)] = f(t, z(t)), \quad t \in J, \end{aligned}$$

and

$$y(0) = -\int_0^T v(s)\,\mathrm{d}s + d + \int_0^T z(s)\,\mathrm{d}s - \int_0^T z(s)\,\mathrm{d}s \leqslant -\int_0^T z(s)\,\mathrm{d}s + d,$$

$$z(0) = -\int_0^T u(s)\,\mathrm{d}s + d + \int_0^T y(s)\,\mathrm{d}s - \int_0^T y(s)\,\mathrm{d}s \geqslant -\int_0^T y(s)\,\mathrm{d}s + d.$$

It shows that y, z are w.c. lower and upper solutions of (1) and the proof is complete. □

Theorem 2. *Put $\lambda = -1$. Let Assumptions H_1, H_3–H_5 hold. Then there exist monotone sequences $\{y_n, z_n\}$ such that $y_n(t) \to y(t)$, $z_n(t) \to z(t)$, $t \in J$ as $n \to \infty$ and this convergence is uniformly and monotonically on J. Moreover, problem (1) has a unique solution $x \in \Delta$ and $x = y = z$.*

Proof. Let

$$y'_{n+1}(t) = f(t, y_n(t)) - M[y_{n+1}(t) - y_n(t)], \quad t \in J, \quad y_{n+1}(0) = -\int_0^T z_n(s)\,\mathrm{d}s + d$$

$$z'_{n+1}(t) = f(t, z_n(t)) - M[z_{n+1}(t) - z_n(t)], \quad t \in J, \quad z_{n+1}(0) = -\int_0^T y_n(s)\,\mathrm{d}s + d$$

for $n = 0, 1, \ldots$. By Lemma 2, we have $y_0(t) \leqslant y_1(t) \leqslant z_1(t) \leqslant z_0(t)$, $t \in J$ and y_1, z_1 are w.c. lower and upper solutions of (1).

Assume that

$$y_0(t) \leqslant y_1(t) \leqslant \cdots \leqslant y_k(t) \leqslant z_k(t) \leqslant \cdots \leqslant z_1(t) \leqslant z_0(t), \quad t \in J$$

for some $k \geqslant 1$ and let y_k, z_k be w.c. lower and upper solutions of (1). Then, by Lemma 2, we get $y_k(t) \leqslant y_{k+1}(t) \leqslant z_{k+1}(t) \leqslant z_k(t)$, $t \in J$. By induction, we have

$$y_0(t) \leqslant y_1(t) \leqslant \cdots \leqslant y_n(t) \leqslant z_n(t) \leqslant \cdots \leqslant z_1(t) \leqslant z_0(t), \quad t \in J$$

for all n. Hence $y_n(t) \to y(t)$, $z_n(t) \to z(t)$, $t \in J$ if $n \to \infty$. Indeed, (y, z) is a solution of the system

$$y'(t) = f(t, y(t)), \quad t \in J, \quad y(0) = -\int_0^T z(s)\,\mathrm{d}s + d,$$

$$z'(t) = f(t, z(t)), \quad t \in J, \quad z(0) = -\int_0^T y(s)\,\mathrm{d}s + d. \tag{4}$$

Note that if we show that $y = z$, then $y = z$ is a solution of (1). Put $P = y - z$, so $P(0) = \int_0^T P(s)\,\mathrm{d}s$, and

$$P'(t) = f(t, y(t)) - f(t, z(t)) = f_x(t, \xi(t))P(t), \quad t \in J,$$

where ξ is between y and z. It yields

$$P(t) = \mathrm{e}^{\int_0^t f_x(s, \xi(s))\,\mathrm{d}s} P(0), \quad t \in J. \tag{5}$$

This and boundary condition for P give

$$P(0) = \int_0^T P(t)\,dt = P(0)\int_0^T e^{\int_0^t f_x(s,\xi(s))\,ds}\,dt.$$

Hence, $P(0)=0$, by Assumption H_5, and finally, $P(t)=0$ on J, by (5). It proves that $y=z$, so $y=z$ is a solution of (1).

To finish the proof it is enough to show that $y=z$ is a unique solution of (1) in Δ. To do it, we need to show that if w is any solution of (1) such that $y_0(t)\leqslant w(t)\leqslant z_0(t)$, $t\in J$, then $y_0(t)\leqslant y(t)\leqslant w(t)\leqslant z(t)\leqslant z_0(t)$, $t\in J$. Assume that for some k, $y_k(t)\leqslant w(t)\leqslant z_k(t)$, $t\in J$. Put $p=y_{k+1}-w$, $q=w-z_{k+1}$. Then

$$p(0)=\int_0^T [w(s)-z_k(s)]\,ds\leqslant 0,\quad q(0)=\int_0^T [y_k(s)-w(s)]\,ds\leqslant 0,$$

and, by Assumption H_3,

$$p'(t)=f(t,y_k(t))-f(t,w(t))-M[y_{k+1}(t)-y_k(t)]\leqslant -Mp(t),\quad t\in J,$$

$$q'(t)=f(t,w(t))-f(t,z_k(t))+M[z_{k+1}(t)-z_k(t)]\leqslant -Mq(t),\quad t\in J.$$

Hence, $y_{k+1}(t)\leqslant w(t)\leqslant z_{k+1}(t)$, $t\in J$. It proves, by induction, that $y_n(t)\leqslant w(t)\leqslant z_n(t)$, $t\in J$ for all n. Taking the limit $n\to\infty$, we get $y=w=z$, so the assertion of Theorem 2 is true. The proof is complete. □

Example 3. Consider the following problem:

$$\text{(c)}\quad \begin{aligned} &x'(t)=e^t\sin^2 x(t)\equiv f(t,x(t)),\quad t\in J=[0,T]\ \text{with}\ T=\ln\sqrt{3},\\ &x(0)=-\int_0^T x(s)\,ds.\end{aligned}$$

It is simple to verify that $y_0(t)=-1$, $z_0(t)=e^t$ on J are w.c. lower and upper solutions of problem (c) and $M=\sqrt{3}$. Moreover $f_x(t,\xi)=e^t\sin 2\xi$, so

$$\int_0^T e^{\int_0^t f_x(s,\xi(s))\,ds}\,dt\leqslant\int_0^T e^{\int_0^t e^t\,ds}\,dt\leqslant\int_0^T e^{t\sqrt{3}}\,dt\approx 0.92<1.$$

Assumptions H_1, H_3–H_5 are satisfied, so problem (c) has in the segment $[y_0,z_0]$ a unique solution, by Theorem 2.

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